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Fundamental Mathematical Theories

George Temple

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Fundamental mathematical theories

By George Temple†
Quarr Abbey, Ryde, Isle of Wight PO33 4ES, UK

The object of this paper is to provide a basic mathematical theory, which will be simple enough to be necessarily self-consistent and sufficiently powerful to yield proofs of the axioms from which the fundamental branches of mathematics are usually derived.

A note on the paper by Martin Hyland‡

This paper contains a collection of George Temple's thoughts about the possibility of recasting the foundations of mathematics. Temple died before he could respond to referees' reports, and the paper as published is not in a definitive state. It is written in an old idiom (effectively one current in Temple's middle years), and a number of points made (for example some remarks about consistency) do not make sense in current terms. Axioms are indicated, but without any explicit formalization, and as a result, it is hard to be sure of the technical content. But it is clear from remarks in the paper that Temple was consciously not formulating his theory within standard formal logic and (rather like Brouwer in his later writings on intuitionism) wished to make a virtue of this.

To the extent that the paper is a throwback to a previous intellectual age, it is probably not worth trying to hold on to all the lines of thought. For example, § 5 interprets propositional logic within the system, but we would not now make much of that. Section 6, on abstract arithmetic, indicates a combinatorial (graphical) description of all countings of a set; the instinct to treat them all seems quite modern, but it is not clear how what follows addresses the representation of mathematical induction within the theory. Section 7 starts with some remarks on what is now called type coercion in theoretical computer science, and models this by a relation between two taxonomic systems; this is an interesting possibility, but the role of the relation in the foundational theory is not apparent.

However, the central idea of the paper, that of a foundational theory based on a single symmetric relation, seems worth further attention. There appears to be no *a priori* reason why such a foundational theory should not be possible; one can certainly code complex structure within a graph. Temple gives no indication of how the symmetric relation (of association) is to be understood; but one intuitive basis for theory might be to imagine a (small) collection of (small) subsets of a set, and to associate just those pairs of elements which both belong to some of these subsets. (Probably Temple had something quite different in mind. Maybe there is a lost intellectual tradition here.)

† Died 30 January 1992.
‡ Department of Pure Mathematics and Mathematical Statistics, 16 Mill Lane, Cambridge CB2 1SB, UK.

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It is surely wise to consider afresh the prospects for a foundational theory based on some relation other than that of membership, but Temple's paper contains enough detail to suggest how one might try to formalize theories along the lines he was considering. Some kind of second order (two sorted) theory is suggested by Temple's distinction between primary and secondary taxa; but the critical ideas about extensions of taxa could be treated in a number of different ways. Unfortunately Temple did not make explicit the basis on which his relation of association is to be extended; nor is it at all clear how the construction of new taxonomic systems from old is to be regarded. So we may never be quite sure as to what the intentions were. But the general project could well be taken further. While Temple did not live to bring his ideas into a state easily accessible to modern readers, he left them something to think about.

1. The ideal theory

(a) *The problem*

The purpose of this investigation into the foundations of mathematics is to carry out the primary part of Hilbert's programme, i.e. to establish the consistency of set theory, abstract arithmetic and propositional logic.

The method by which we propose to achieve this result is to construct a new and fundamental theory from which these mathematical theories may be deduced.

Such a theory should be simple enough to exhibit its self-consistency and powerful enough to provide proofs of Zermelo's axioms for set theory, Peano's axioms for arithmetic and the Whitehead–Russell axioms of inference. It must necessarily be independent of any existing theory, and of any existing system of axioms and definitions. The construction of such a theory can carry no guarantee that it will issue in any of the existing theories. The whole process is therefore essentially a philosophical experiment, whose value is to be judged by its success in validating current mathematical disciplines.

Everything in this new theory is to be sacrificed to ensure that it is self-consistent, and can be seen to be self-consistent. This overriding principle enables us to formulate the general characteristics of such an 'ideal' theory.

In general, a mathematical theory consists of (i) its material, i.e. certain undefined objects, which are characterized by (ii) its form, i.e. certain undefined relations, which give some indication of the nature of the objects, and (iii) its axioms, which are the basic, unproved propositions restricting the operations of the relations.

(i) In order to obtain a self-consistent theory it is clearly advisable to begin experimenting with the simplest type of theory, namely a 'single-sorted' theory in which there is only one species of undefined relations (for example, we should exclude the type of geometric theory with several undefined objects, such as 'points', 'lines' and 'planes', and with several relations and axioms of 'incidence').

(ii) In the interests of simplicity it seems advisable to start with a 'symmetric' relation, rather than an 'asymmetric' relation and to consider only a symmetric relation between pairs of terms, rather than between larger collections. For an asymmetric relation, R, between objects p and q, in that order (i.e. pRq) is distinct from the relation between the objects q and p, in the reverse order (i.e. qRp). Hence an asymmetric relation involves the concept of 'order', and is thus more complex than any symmetric relation. Also, it is manifestly simpler to consider a symmetric relation,

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such as 'equivalence', between pairs of objects, rather than a symmetric relation, such as 'collinearity', between triads of points.

The restrictions to symmetric relations has a further great advantage, in that it enables us to explain what is involved in such a concept. So far as mathematical theory is concerned, a symmetric relation between pairs of objects simply means that all *pairs* of objects in our theory can be divided into two collections, A and D, such that any pair in A are 'related' and any pair in D are 'unrelated'. It is not necessary to characterize a symmetric relation any further, although we can give numerous examples of different kinds of symmetric relation (see § 1 *b*).

(iii) We are thus led to experiment with a theory which consists of certain undefined objects, certain pairs of which are 'related'. We have, in fact, thus enunciated an axiom of our theory, namely that any pair of objects are either 'related' or 'unrelated'.

The foundation of our theory, as we have so far constructed it, consists simply of some undefined objects, a single undefined relation, and only one axiom. These relations may appear somewhat sparse and exiguous, but, in fact, they are complete and sufficient to define a 'taxonomic system'.

The foundations incorporate only one type of relation, and contain only one type of object. The question of consistency does not arise, since there is only one axiom. The sufficiency of the foundation remains to be demonstrated by exploring what can be developed from such an exiguous basis, and this is the subject of this paper.

The essential characteristic of the taxonomic theory is that interest is concentrated, not on individual objects (such as Euclidean points) but on pairs of taxa. This makes the theory 'self-contained' and free from any reference to 'properties' of individual terms (such as the 'individuality' of points or the 'truth of propositions').

A similar powerful method has been employed by Nash-Williams. In his theory, the basic elements are directed pairs of elements, p, q, which form the vertices of a 'digraph'. The relation $p \rightarrow q$ can then be regarded as an abstract form of 'inclusion', in which the set p is included in the set q.

This method provides an illuminating representation of set theory, but it does use the concept of order to distinguish the relation $p \rightarrow q$ from the relation $q \rightarrow p$ and thus is basically distinct from taxonomic theory.

In our theory a taxonomic system is indeed an abstract graph, but the connections between 'vertices' define associative pairs of taxa and not the asymmetric relation of inclusion.

(*b*) *Consistency and development*

The preceding examples illustrate the consistency of a taxonomic system specified by relations of association.

Thus, a plane graph is constructed by scattering a number of 'dots' (i.e. vertices) over a sheet of paper, and by joining certain pairs of dots by lines to indicate their association. The choice of which pairs are to be associated or dissociated is entirely arbitrary, and such choices, as expressed by the propositions, $p \sim q$ or $r \not\sim s$, are entirely independent of one another.

Similarly, when we specify a matrix by requiring the entries to be either **1** or **0**, our choice of these entries is again entirely arbitrary, provided, of course, that they make the matrix symmetric.

A taxonomic system, specified by a collection of taxonomic relations of association and dissociation, is essentially self-consistent, but this is not the only condition which it must satisfy.

The success of a taxonomic theory must also depend on its powers of generating taxa, analogous to the 'unions' and 'intersections; of set theory. The possibility of this development requires that, in a taxonomic system, there must be some distinction between the 'data' – the taxa which are initially given – and the 'quaesita' – the taxa which are to be constructed from the data.

(c) *Primary and secondary taxa*

We therefore distinguish between the 'primary' taxa and the 'secondary' taxa. The former are explicitly specified by their mutual taxonomic relations. Thus, a primary taxon p is characterized by its 'neighbourhood', $N(p)$, the collection of primary taxa associated with p.

The 'secondary' taxa are given indirectly by their mode of construction from the primary taxa. It will appear (§ 2 *h*) that this determines the taxonomic relation of each secondary taxon x to each of the primary taxa, and the mutual taxonomic relations of the secondary taxa.

Thus, the information provided by a taxonomic system above the primary taxa is summarized by their neighbourhoods, and it is implicit that any pair of primary taxa p and q are distinct and that therefore they have different neighbourhoods $N(p)$ and $N(q)$.

This distinction can be expressed in the form that, if p and q are any primary taxa, then there is another primary taxon s, such that $p \sim s$ and $q \not\sim s$, or $p \not\sim s$ and $q \sim s$. For any taxonomic system T its primary taxa by themselves will be described as its nucleus T^1.†

This principle excludes from consideration such taxonomic systems as that represented by the relation,

$$p \sim q \sim r \sim p,$$

in which all the neighbourhoods would be $\{p, q, r\}$. This is reminiscent of Pauli's exclusion principle in quantum theory.

Thus, the primary taxa are all different, but it will appear (§ 2 *k*) that a pair of 'secondary' taxa, x and y, although constructed in different ways, may have identical neighbourhoods.

Hence we shall write

$$x = y \quad \text{if} \quad N(x) = N(y).$$

The distinction between primary and secondary taxa is the basis of the taxonomic analogue of Frege's theory of identity, which appears later in § 2 *j*.

(d) *Sub systems*

Independently of the division of the collection of pairs in a taxonomic system T into the associated and dissociated pairs, it is possible to select from T a smaller system S, each taxon in which is a member of T; e.g. we can take S to be the taxa in the neighbourhood $N(p)$ of a specified taxon p of T.

† It is evident from the subsequent development of Professor Temple's theory that the association $p \sim p$ of each primary taxon with itself was taken by him as a necessary characteristic of the symmetric relation of association.

Then it is almost trivial to observe that such a collection S is also a taxonomic system, and can be described as a 'sub-system' of T.

(*e*) *Null taxa*

The simplest examples of the 'creation' of new taxa to be added to a given taxonomic nucleus are the 'null' taxa, which correspond to Zermelo's 'empty' sets, and the 'total' taxa, which have no analogue in set theory.

A 'null' taxon, $\emptyset$, of a system T is defined to be a taxon which is dissociated from all the taxa in the nucleus T^1 of T, and is dissociated from $\emptyset$ itself, i.e.

$$\emptyset \not\sim s \quad \text{for all} \quad s \in T^1$$

and

$$\emptyset \not\sim \emptyset.$$

Any taxa in T^1 which is not null will be called a 'real' taxon. Hence, if p is a real taxon, there are some taxa s in T^1 such that $p \sim s$, and, in particular, $p \sim p$. Also, if x is any primary taxon such that $x \not\sim x$, the x is a null taxa, and $x = \emptyset$.

In the graphical representation of a taxonomic system, the null taxa are represented by isolated vertices.

The definition of null taxa is important, not only because it enables us to dispense with Zermelo's axiom of the empty set, but because it shows that the taxonomic theory provides a 'home' for the 'non-existent' objects studied by Meinong which have 'being' but no existence.

Null taxa are secondary taxa, and their 'equivalence' requires separate consideration. However, if there were two null taxa, $\emptyset_1$ and $\emptyset_2$, we should have that

$$\emptyset_1 \not\sim s \quad \text{and} \quad \emptyset_2 \not\sim s$$

whence $N(\emptyset_1) = N(\emptyset_2)$ and so $\emptyset_1 = \emptyset_2$.

This justifies us in describing all null taxa as 'equivalent'.

(*f*) *Total taxa*

In complete contrast to the null taxa, we can define the 'total' taxon, τ, of the nucleus of a system T, to be the unique taxon τ such that τ is associated with all the primary taxa of T,

$$\tau \sim s \quad \text{for all} \quad s \in T^1,$$

while we respect the isolation of the null taxon by adding the condition that $\tau \not\sim \emptyset$.

The total taxon τ of a system T can be regarded as the 'characteristic taxon' of T in the sense implied by Zorn's lemma and the axiom of choice.

The total taxon, τ, is a secondary taxon and its existence in the taxonomic system corresponds to the existence of 'maximum' set in Zorn's principle.

(*g*) *Axiom of choice*

In set theory the axiom of choice can be obtained as an immediate consequence of Zorn's lemma, but in taxonomic theory there is a further matter needing consideration. Corresponding to each system in a collection of taxonomic systems $T_1, T_2, \ldots$ there is a total taxon $\tau_1, \tau_2, \ldots$, but these total taxa will not form another taxonomic system A, unless they are provided with appropriate taxonomic relations.

We therefore restrict ourselves to the consideration of a collection of systems $T_1, T_2, \ldots$, which are sub-systems of one and the same system T. We can then relate the total taxa, $\tau_1, \tau_2, \ldots$ of these taxonomic systems by means of the condition that

$$\tau_j \sim \tau_k$$

if there is a taxon s in each of the sub-systems T_j and T_k. The total taxa then become a taxonomic system A, which contains one taxon, τ_j from each sub-system T_j.

(*h*) *Power systems*

A further illustration of the expansion of a taxonomic system is provided by the existence of 'power systems'.

In set theory the 'power' set of a given set S is the collection of all the subsets of S, and analogous 'power' systems of a given taxonomic system T can be defined in many ways as collections of various types of sub-systems of T.

The simplest sub-systems of T are the secondary taxa defined as the pairs of associated primary taxa of T

$$X = \{p, q\}, \quad Y = \{r, s\}, \ldots,$$

with the conditions that $p \sim q, r \sim s, \ldots$.

These new taxa can be added to the primary taxa of T to form the primary taxa of a 'power' system T^* by imposing the relations

$$\{p, q\} \sim p, \quad \{p, q\} \sim q,$$

$$\{p, q\} \sim \{r, s\} \quad \text{if} \quad p \sim r, q \sim s.$$

We then obtain the nucleus of a taxonomic system T^* which includes the nucleus T^1 of the original system T, with the crucial property that T^1 is a proper part of T^*.

(*i*) *The axiom of infinity*

The process by which the nucleus of the power system T^* has been constructed on the basis of the original nucleus of T can be repeated to construct the power system T^{**} of T^*, and to ensure that T^* is a proper part of T^{**}.

This process can be repeated indefinitely to form an infinite sequence of systems. We have thus established the existence of an infinite system and have rescued mathematical analysis, which is in fact the theory of infinite systems.

(*j*) *Sets and taxa*

It is a curious and unexpected result that the search for a fundamental and primitive mathematical theory should lead to the discovery of a new relation unknown to set theory, namely the taxonomic relation of 'association'. It is this relation which gives to the new theory its variety, its unity and its power.

All 'sets' have the same structure. A set is just a collection of elements, each with the same property which plays no further part in the theory. The sets differ only in the number of elements which they include.

In a taxonomic system of N primary taxa, there are $\frac{1}{2}N(N-1)$ pairs of different taxa, and the number of associated taxa may be 1, 2, 3, or any number up to $M = \frac{1}{2}N(N-1)$, whence the number of different systems, each of N taxa, is at least

$$1 + 2 + 3 + \ldots + \tfrac{1}{2}N(N-1) = \tfrac{1}{2}M(M+1),$$

not counting the various kinds of interconnection.

This enormous variety greatly adds to the interest of compiling what Listing (1861) would have called a taxonomic 'census'.

At the same time, the relation of association exerts a restraining and directing influence in the description of such taxa as sub-systems, serial relations and atomic taxa. In taxonomic theory it is not sufficient to assume the existence of these concepts, but it is necessary to show that they are constructed solely in virtue of the relation of association. For example, it is not sufficient to assert the existence of a one-to-one 'correspondence' between the taxa of two systems. The taxonomic theory demands an analysis of such a morphism into a scheme of explicit associations.

So much has been claimed above for taxonomic theory that it is desirable to recognize explicitly that it has no claim to be the only fundamental theory of mathematics.

For example, an abstract geometry can be based on the undefined relation of 'collinearity' between three undefined points; abstract group theory on the operation $X \cdot A$ by which an (undefined) operator X converts an operator A into the operator $X \cdot A$; category theory on an abstract concept of the functional relation and the axiomatic set theory of Nash-Williams is based on the theory of directed graphs.

Taxonomic systems are just one example of a fundamental theory which seems to be worth study because of its simplicity and power.

(*k*) *Sources*

The present paper, which discusses 'sets' without 'elements', has been greatly influenced by the studies of 'geometry' without 'points', published in the early part of the present century.

The classical examination of the axioms of Euclidean geometry made by Hilbert in 1902, and A. N. Whitehead's Cambridge tracts on projective and descriptive geometry had accepted points, lines and planes as undefined terms, but, by 1914, Whitehead and Russell had envisaged the possibility of eliminating points from the primary undefined terms of geometry.

In his first scheme of relativistic cosmology, Whitehead made the fundamental relation to be 'extensive abstraction' with the 'inclusion' of one 'event' by another. But the work of Theodore de Laguna suggested to him that the basic connection should be 'extensive connection', by which one event 'overlapped' another.

Meanwhile, Lesniewski had independently devised a similar theory based not on overlapping, but on the complementary relation of discreteness or dissociation. This work first became accessible in the West in the accounts given by Tarski and by Leonard & Goodman.

The theory of 'association', given in this paper, owes much to these researchers, which it has developed so as to include much of Hilbert's original 'programme'.

But, above all, tribute must be paid to the pioneer work of Bertrand Russell, and his exposition of the researches of Frege and Peano.

2. Taxonomic algebra

(*a*) *The need for algebra*

In 'naïve' set theory, as it has been happily named by Halmos, a 'set' is a collection of 'elements', which 'belong' to the set, inasmuch as they possess the same 'property'. The problem of set theory has always been to provide some definitions of these terms, which would eliminate the notorious paradox noted by Cantor and Russell.

In taxonomic theory the difficulty is met by eliminating the concept of 'element' from the list of undefined terms, and by characterizing 'taxa' (instead of sets) by their mutual relations of association and dissociation. This radical change encourages the hope that no more undefined terms will be needed and that the subsequent definitions may be expressed entirely in terms of the taxonomic relation.

This ambitious programme implies that an attempt must be made to define all technical terms, including the grammatical connectives, 'and', 'or', instead of giving illustrations of their use. This is a semantic problem of what may be called a 'species of radical definition'. It will conduce to clarity and conciseness to express our solution in the symbolism of binary algebra.

To achieve this result and to define all the algebraic concepts, usually expressed by the symbol $>$, $=$, $-$, $\times$, solely in terms of association and dissociation, requires a preliminary semantic analysis.

(*b*) *The algebra of association*

To equip a taxonomic theory with an algebra based on the relation of 'association', we begin by noting that, not only is the collection of pairs of taxa (p, q) divided into two disjoint classes – the associated and dissociated pairs – but it is also possible to introduce a minimal kind of 'order' by defining that

(i) any associated pair 'precedes' any dissociated pair, and
(ii) all associated pairs are 'equipollent'
(iii) all dissociated pairs are 'equipollent'.

We express this definition by writing
(i) $(p, s) > (q, s), (q, s) < (p, s) \quad \text{if} \quad p \sim s, q \not\sim s.$
(ii) $(p, s) = (q, s) \quad \text{if} \quad p \sim s, q \sim s.$
(iii) $(p, s) = (q, s) \quad \text{if} \quad p \not\sim s, q \not\sim s.$

Since $p \sim p$ and $p \not\sim \emptyset$, p being any primary taxon, we can introduce the algebraic elements $\mathbf{1}$, $\mathbf{0}$, by the definition

$$\mathbf{1} = (p, p), \quad \mathbf{0} = (p, \emptyset).$$

Hence

$$(p, s) = \mathbf{1} \quad \text{if} \quad p \sim s, \quad (p, s) = \mathbf{0} \quad \text{if} \quad p \not\sim s.$$

We also note the familiar relations of binary algebra:

$$\mathbf{1} = \mathbf{1}, \quad \mathbf{0} = \mathbf{0}, \quad \mathbf{1} > \mathbf{0}.$$

(The symbols $\mathbf{1}$, $\mathbf{0}$, for the algebraic elements will be printed in bold type to distinguish them from the arithmetical unity and zero.)

Finally, we briefly define that

$$(p, s) \geqslant (q, s)$$

if (q, s) does not 'precede' (p, s). This implies the relations $(p, s) = (q, s), (p, s) > (q, s)$, but avoids the use of the connective 'or'.

(*c*) *Maxima and minima*

The next step is to establish the apparently trivial result that any collection of algebraic elements $\{(p, q)\}$ possesses a maximum and a minimum.

Such a collection is either homogeneous or heterogeneous.

If it is homogeneous, then

$$(p, q) = (x, y)$$

for all elements in the collection. Then any element in the collection is at once a maximum and a minimum.

If the collection is heterogeneous, then it must contain elements $(p,q),(x,y)$ such that

$$(p,q) > (x,y).$$

Then (p,q) is a maximum element, and (x,y) is a minimum element, and

$$(p,q) = \mathbf{1}, \quad (x,y) = \mathbf{0}.$$

(*d*) *Sums and products*

We can now define the 'sum' and 'product' of any collection of elements, without the use of the connectives, 'and', 'or'.

We define the 'sum' ρ and the 'product" σ of the taxa p,q in terms of the taxonomic relations connecting p,q with any taxon s. Thus

$$(\rho,s) = \max\{(p,s),(q,s)\}, \quad (\sigma,s) = \min\{(p,s),(q,s)\}$$

Finally, we define the taxa ρ and σ to, respectively, the 'union' and the intersection of the taxa p.

If the collection consists of only two elements,

$$(p,s), \quad (q,s)$$

then, the definitions take the colloquial forms

$$\rho \sim s \quad \text{if} \quad p \sim s \quad \text{or} \quad q \sim s,$$

$$\sigma \sim s \quad \text{if} \quad p \sim s \quad \text{and} \quad q \sim s.$$

As usual we shall write

$$\rho = \sum p, \quad \sigma = \prod p;$$

and for the collection of taxa, p,q,

$$\rho = p+q, \quad \sigma = p \times q \quad \text{or} \quad p \cdot q.$$

We note that since $\mathbf{1}\times\mathbf{1} = \mathbf{1}$ and $\mathbf{0}\times\mathbf{0} = \mathbf{0}$, hence

$$(\rho,s)\cdot(q,s) = (q,s)$$

and

$$(\sigma,s)\cdot(p,s) = (p,s)\cdot(q,s)\cdot(p,s) = (\sigma,s).$$

Therefore

$$(\sigma,s) \leqslant (\rho,s).$$

(*e*) *Conjugate elements*

In defining the 'complement' of a taxon it proves to be convenient to introduce the 'conjugate' of an element by the definition

$$\mathbf{1}' = \mathbf{0}, \quad \mathbf{0}' = \mathbf{1},$$

whence $(p,s)'$ is the conjugate of (p,s) if

$$(p,s)' = \mathbf{1} \quad \text{if} \quad (p,s) = \mathbf{0},$$

and

$$(p,s)' = \mathbf{0} \quad \text{if} \quad (p,s) = \mathbf{1}.$$

(f) Inclusion

The relation of 'inclusion' seems to be ineluctably spatial, but an entirely abstract definition has been given by Theodore de Laguna, Steven Lesniewski, and A. N. Whitehead.

In the terminology of taxonomic theory, Lesniewski defined that a taxon x is 'included' in a taxon p if any taxon s dissociated from p is also dissociated from x.

Similarly, Whitehead defined that a taxon x is 'included' in a taxon p if any taxon s associated with x is also associated with p.

Both definitions can be combined in the single relation,

$$(x, s) \leqslant (p, s),$$

and we then write $x \subseteq p$.

This relation can be put in the equivalent forms

$$x = p \cdot x \quad \text{and} \quad p = p + x.$$

For

$$(xp, s) = (x, s) \cdot (p, s) \leqslant (x, s)$$

and

$$(x, s) = (x, s)(x, s) \leqslant (x, s)(p, s) = (xp, s).$$

Hence

$$(x, s) = (xp, s).$$

Also, if

$$(x, s) = (xp, s) = (x, s)(p, s),$$

then

$$(x, s) \leqslant (p, s).$$

In order to justify the abstract definition of inclusion, we give a selection of the properties of the relation, which show how it embodies a number of familiar concepts.

(i) Two taxa, x and y, are associated only if there is a non-null taxon z such that

$$z \leqslant x, \quad z \leqslant y.$$

For, if $(x, s) \geqslant (z, s)$ and $(y, s) \geqslant (z, s)$, then

$$(x, z) \geqslant (z, z) = 1,$$

and

$$(y, z) \geqslant (z, z) = 1.$$

Hence $x \sim z$ and $y \sim z$. Conversely, if $x \sim y$, and $z = x \cdot y$, then

$$(z, x) = (x, x)(y, x) = (y, x) = \mathbf{1}.$$

Hence z is not null, and $z \leqslant x$, $z \leqslant y$.

(ii) If $p \geqslant q$ and $q \neq \emptyset$, then $p \sim q$. For $(p, s) \geqslant (q, s)$, and $(p, q) \geqslant (q, q) = \mathbf{1}$.

(iii) If $p \geqslant q$ and $p \not\sim q$, then $q = \emptyset$. For $\mathbf{0} = (p, q) \geqslant (q, q)$, whence $q \not\sim q$ and $q = \emptyset$.

(iv) If x and y are dissociated, and $u \leqslant x$, $v \leqslant y$, then $u \not\sim v$.

(v) If p and q are dissociated and $x \leqslant p$, $x \leqslant q$, then $x = \emptyset$.

(vi) If $p \not\sim q$ (and $p \neq \emptyset$), then $pq = \emptyset$, and conversely if $pq = \emptyset$, then $p \not\sim q$. Conversely, if $pq = \emptyset$, then since $(pq, p) = (q, p)$, we have $p \not\sim q$.

(vii) Finally, we note the formulae

$$p + p = p, \quad p \cdot p = p,$$

whence, if $\sigma = pq$, then $\sigma \cdot p = \sigma$, and σ, the intersection of p and q, is included in p (and q).

(*g*) *Complements*

In framing the definition of the intersection, σ, of the taxa p, q, we have, as it were, subtracted σ from the taxa p and q, for $\sigma = pq$, and $pq \subset p$. We must now consider what 'remainder' is left. We therefore prove that there is a unique taxon x such that

$$p = \sigma + x \quad \text{and} \quad \sigma x = \emptyset.$$

To prove this we note that the only cases for consideration are the following:

(i) $$(p, s) = \mathbf{0}, \quad (\sigma, s) = \mathbf{0};$$

(ii) $$(p, s) = \mathbf{1}, \quad (\sigma, s) = \mathbf{0};$$

(iii) $$(p, s) = \mathbf{1}, \quad (\sigma, s) = \mathbf{1}.$$

Then (x, s) is uniquely defined as $(x, s) = \mathbf{0}$ in case (i), $(x, s) = \mathbf{1}$ in case (ii), and $(x, s) = \mathbf{0}$ in case (iii), in order to satisfy the conditions $(p, s) = (\sigma, s) + (x, s)$ and $(\sigma, s)(x, s) = \mathbf{0}$.

Similarly, we can prove that there is a unique taxon y such that

$$q = \sigma + y \quad \text{and} \quad \sigma y = \emptyset.$$

We can now define the complement $\overline{p}$ of a primary taxon p as the union of all the primary taxa q dissociated from p, together with all the 'remainders' y when $pq = \sigma$ is subtracted from q.

Then, if we sum all the relations

$$q = \sigma + y, \quad \sigma y = \emptyset,$$

we obtain

$$\sum q = \tau \sum \sigma = p \sum q = p\tau = p, \quad \sum y\sigma = \sum ypq = p \sum yq = p \sum y,$$

and the complement $\overline{p}$ of p is $\sum y$. Hence,

$$\tau = \overline{p} + p \quad \text{and} \quad p\overline{p} = \emptyset.$$

From the symmetry of these results it follows that

$$\tau = \overline{\overline{p}} + \overline{p} \quad \text{and} \quad \overline{p}\overline{\overline{p}} = \emptyset,$$

if $\overline{\overline{p}}$ is the complement of $\overline{p}$. However, we have proved that the complement of p is the unique taxon $\overline{p}$, whence $\overline{\overline{p}} = p$. If $z = xy$, we can prove that $\overline{z} = \overline{x} + \overline{y}$.

For $xy\overline{x} + \overline{y} = \emptyset$, and

$$xy = (\overline{x} + \overline{y}) = xy + \mathbf{1} - x + \mathbf{1} - y = 1 + (1 - x)(1 - y) = 1,$$

since $[1 - (x, s)] \cdot [1 - (y, s)]$ is either $\mathbf{0}$ or $\mathbf{1}$. Similarly, if $z = x + y$, then $\overline{z} = \overline{xy}$.

(*h*) *Secondary taxa*

The operation of forming the sum or product or complement can be repeated so as to form taxa of the form

$$a = (p + q)r \quad \text{and} \quad b = (p \cdot r) + (q \cdot r),$$

Table 1.

(p,s)	(q,s)	(r,s)	(a,s)	(b,s)
0	0	0	0	0
1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	1	1	1	1
1	0	1	1	1
1	1	1	1	1

and the corresponding algebraic elements (a,s) and (b,s) can be calculated for all possible values of the elements $(p,s),(q,s),(r,s)$ as in table 1. It then appears that $(a,s)=(b,s)$ for all primary taxa s.

The taxa which can be constructed by repeated operations of summation, multiplication and complementation are 'secondary taxa'. Although no two primary taxa are identified, the preceding example shows that there will exist secondary taxa, such as a and b, for which

$$(a,s)=(b,s)$$

for all primary taxa s. Such secondary taxa we shall describe as 'equivalent', and we shall write $a=b$.

It is easy, if tedious, to show that, according to this definition, the operations of addition and multiplication are associative, commutative and distributive.

Also, it is a matter of calculation to ascertain if there is a primary taxon s associated with each of a pair of secondary taxa a and b. If such a taxon s exists we shall say that a and b are 'associated'.

(*i*) *Neighbourhoods*

The definition of the 'neighbourhood', $N(p)$, of any primary or secondary taxon p, has been given as the collection of all primary taxa s which are associated with p, i.e. in the usual notation (§ 1 *d*):

$$N(p)=\{x; x\sim p\}.$$

Then the definition of 'equivalence' can be put in the form

$$p=q \quad \text{if} \quad N(p)=N(q),$$

i.e. if the neighbourhoods of p and q are identical, and the association of $N(p)$ and $N(q)$ can be defined as

$$N(p)\sim N(q)$$

if there exists a primary taxon s in both $N(p)$ and $N(q)$.

Since we have equipped neighbourhoods with a relation of association, we can construct a 'power' system in which the primary taxa are the neighbourhoods of the primary taxa in the given taxonomic system T.

(j) 'Sinn' and 'Bedeutung'

In the work of Frege the concepts of 'Sinn' and 'Bedeutung' are developed for propositions, but it is possible to provide an analogous theory for taxa.

We therefore define the 'essence' (or Sinn) of any taxon x to be its neighbourhood $N(x)$, i.e. the collection of all primary taxa associated with x, and we define the 'representations' (Bedeutung) of x to be all the taxa with the same neighbourhood as x.

Thus all representations of a taxon x are equivalent.

Straightforward calculation with the appropriate algebraic elements will verify the following examples:

(i) any taxon x has (among others) the representations

$$x = xy + x\overline{y}$$

$$x = (x + y) \cdot (x + \overline{y}),$$

for

$$(x + y) \cdot (x + \overline{y}) = xx + x\overline{y} + xy + y\overline{y} = x + x + \emptyset = x;$$

(ii) equivalent representations are

$$(p + q) + r = (p + r) + q = \overline{a \cdot b}$$

where a and b are the complements of $p + q$ and r respectively;

(iii) similarly,

$$\overline{p \cdot q} = \overline{p} + \overline{q}.$$

(k) Taxonomic 'proof theory'

It is desirable to draw attention to the type of reasoning which has been used in establishing the results of the preceding sections. There has been no appeal to the axioms of propositional logic or the standard rules of inference. Indeed, this would have been illegitimate since we have not yet introduced any fundamental theory of deduction. We have relied entirely on 'immediate deduction', or the analysis of our definitions, and on the elementary methods of 'complete induction'.

What has made this latter method possible is the morphism which exists between a system of primary taxa and a binary algebra. Thus, to any primary taxon x there corresponds the algebraic elements (x, s), and to any operation on primary taxa there corresponds an operation in the algebra. Thus $(p, s) + (q, s)$ corresponds to $p + q$, and $(p, s) \cdot (q, s)$ to $p \cdot q$. Hence, any theorem in a taxonomic system is represented by a theorem in the binary algebra. Since the latter contains only two species of elements, namely **1** and **0**, the corresponding algebraic theorem can be established by enumerating all possible cases and exhibiting their validity. Even when the number of taxa is infinite, their relations fall into a small number of groups which can be examined completely, and, the hitherto denigrated method of 'complete induction' becomes an indispensable method of investigation.

3. Connectivity

(a) Higher order neighbourhoods

The global structure of a taxonomic system can be investigated by generalizing the concept of a 'neighbourhood'.

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The neighbourhood, $N(p)$, of a primary taxon p has been defined (§ 1 d) as the collection of the primary taxa x which are associated with p, i.e.

$$N(p) = \{x; x \sim p\}.$$

Regarding $N(p)$ as the 'first-order neighbourhood', $N^{(1)}(p)$ of p, we can proceed to define the 'second-order neighbourhood', $N^{(2)}(p)$, as the collection of all primary taxa y which are each associated with some taxon s of $N^{(1)}(p)$, i.e.

$$N^{(2)}(p) = \{y; y \sim x, \quad x \in N^{(1)}(p)\}.$$

This process can be repeated to yield the 'third-order neighbourhood', and, thereafter, to generate a sequence of 'higher-order neighbourhoods', such that any neighbourhood, $A(p)$, in the sequence is succeeded by a neighbourhood $B(p)$, such that

$$B(p) = \{z; z \sim y, y \in A(p)\}.$$

Two possibilities present themselves:

(i) The sequence of neighbourhood may terminate with a neighbourhood, $M(p)$, which is identical with its succeeding neighbourhood, $S(p)$, so that, if $x \in S(p)$, then $x \in M(p)$.

(ii) The sequence of neighbourhoods may not terminate.

(b) *Finite systems*

If the sequence of neighbourhoods terminates for all initial neighbourhoods, $N(p)$, (i.e. for each primary taxon p), then we have another characteristic of a 'finite' system.

In such a system there can be no unending sequence of primary taxa, and hence it is finite in terms of the preceding definition (§ 1 k).

It is desirable to show that there are systems, which are finite in accordance with the new definition, without recourse to any arithmetical concept. We therefore consider a 'cyclic' taxonomic system, such as

$$p \sim q \sim r \sim s \sim p.$$

Then

$$N^{(1)}(p) = \{s, p, q\}, \quad N^{(2)}(p) = \{r, s, p, q, r\} = N^{(3)}(p).$$

In general a cyclic system is such that

(i) any taxon x in the system has a successor y such that $x \sim y$,

(ii) the sequence of neighbourhoods, with the initial neighbourhood $N(y)$, has the terminal neighbourhood $N(x)$ and y is the successor of x. In such a cyclic system there are no unending sequences, so that the original definition of a finite system is satisfied.

If the sequence of neighbourhoods does not terminate for some initial neighbourhood, then the system is certainly not finite, but its species of infinitude needs examination in terms of the 'connectivity' of the system, which we shall proceed to analyse in § 3 d.

However, if the sequence of neighbourhoods does not terminate we can still form the taxonomic 'sum', $\Omega(p)$, of all the higher order neighbourhoods $R(p)$ of p.

(c) *Closure*

If $R(p)$ is a terminal neighbourhood of a primary taxon p, then

$$R(p) = \{x; x \sim y, \quad y \in R(p)\}.$$

Hence, if $x \in R(p)$ and $z \notin R(p)$, then $x \not\sim z$, i.e. any taxon x in $R(p)$ is dissociated from any taxa z not in $R(p)$.

Such a collection as $R(p)$ will be said to be 'closed'.

If $\Omega(p)$ is the 'sum' of all the higher order neighbourhoods of $R(p)$ of p, we can establish a similar result.

For, if $x \in \Omega(p)$ and $z \notin \Omega(p)$, then there must be some neighbourhood, $R(p)$ (in the sequence which starts with $N(p)$) such that

$$x \in R(p) \quad \text{and} \quad z \notin R(p).$$

Hence, as before, $x \not\sim z$, and, therefore $\Omega(p)$ is closed.

(*d*) *Disjoint closed collections*

We can now show that any taxonomic system T is composed of disjoint, closed collections.

If J and K are any pair of different, closed collections in T, and they are not disjoint, then there is a taxon x such that

$$x \in J \quad \text{and} \quad x \in K.$$

But, since J and K are different collections, there must be a taxon y, such that

$$y \in J \quad \text{and} \quad y \notin K.$$

Now, x and y are both in J, whence

$$x \sim y.$$

And, since $x \in K$ and $y \notin K$,

$$x \not\sim y.$$

Hence there is no such taxon in x, and therefore J and K are disjoint.

Now consider all the terminal neighbourhoods, $R(p)$, which terminate a sequence of higher order neighbourhoods, and all the quasi-terminal neighbourhoods, such as $\Omega(p)$.

Any pair of these terminals, or quasi-terminal neighbourhoods are either identical or different. If they are different, we have just proved that they are disjoint.

Now any primary taxon p of T is composed of different, closed collections, some of which may be finite, and some of which may be infinite. This completes the analysis of the concept of 'infinite' systems, to which we referred in § 3 *b*.

4. Atomic taxa

(*a*) *Holism and atomism*

The difference between set theory and taxonomic theory may be compared with the difference between 'atomistic' and 'holistic' theories of the foundations of geometry.

In atomistic theories 'points' are taken as primary and indivisible units, from which lines, surfaces etc. are to be constructed. In holistic theories it is solid bodies, extended in space, which are taken to be primary elements, and it is the points which are to be constructively defined.

Similarly, in set theory it is the 'elements' which are taken as primary and indivisible units, and 'sets' and 'sub-sets' which are taken to be collections of elements. In taxonomic theory it is the 'taxa' which are fundamental, while 'atomic' taxa have to be constructed.

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Criticism of set theory may be interpreted to mean that set theory does not provide any implicit relation by which points or elements are to be combined into solid bodies or sets respectively (Wittgenstein 1922).

It is primarily to A. N. Whitehead and Bertrand Russell that we owe a 'holistic' theory, for our analysis of primary taxa into atomic taxa is a development of the method of 'extensive' abstraction.

Our problem is to provide a definition of 'atomic' taxa such that any taxon can be shown to be the sum of all the indivisible atomic taxa which it includes.

(b) *Clusters*

To obtain a constructive definition of atomic taxa, we begin with a preliminary description of an atomic taxon x as an 'indivisible' unit, i.e. such that it is not the sum of two disjoint non-null taxa.

Now any taxon x can be expressed in the form

$$x = xp + x\overline{p},$$

where the two parts xp and $x\overline{p}$ are disjoint, since no primary taxon is included in both xp and $x\overline{p}$.

This bisection of x fails only if one of the pairs xp or $x\overline{p}$ is null. Hence if x is indivisible, then for any p either xp or $x\overline{p}$ is null and thus $x = x\overline{p}$ or $x = xp$, i.e. $x \subset \overline{p}$ or $x \subset p$.

This suggests that an indivisible taxon x might be located by enumerating all the primary taxa or their complements in which it is included. But it is easily seen that it is not necessary to consider all the primary taxa which include x; only a selection of such taxa is required.

For, if $x \subset p$ and $x \subset q$, then $(x, s) \leqslant (p, s)$ and $(x, s) \leqslant (q, s)$, whence $x \sim p$ and $x \sim q$, only if $p \sim q$, and if $p \not\sim q$ then $x \not\sim p$ and $x \not\sim q$.

It should, therefore, be sufficient to enumerate only those taxa which include x, and which are associated with one another. Such a collection of taxa we shall call a 'cluster' C, with the definition that $p \subset C$ and $q \subset C$, if and only if $p \sim q$.

Any cluster C should serve to provide an approximate location for an individual taxon but to obtain the greatest precision the number of taxa in the cluster should be clearly as large as possible.

Now a cluster C will be as large as possible when no further taxon can be added to C without destroying its essential characteristic, and, if a primary taxon r is not in C, then it must be dissociated from some taxon p in C. We therefore define a 'complete' cluster K by the condition that

(i) if $p, q \in K$, then $p \sim q$, and
(ii) if $r \notin K$, then there exists some p in K, such that

$$r \not\sim p.$$

(c) *Fractal construction of clusters*

Although we cannot construct an individual cluster by means of a sequence we can construct, simultaneously, all clusters in a taxonomic system by means of a fractal graph. This is possible because, from any incomplete cluster X we can simultaneously construct all the clusters

$$Y_1, Y_2, \ldots,$$

each of which Y_k consists of the taxa in X and some taxon q_k not in X, but associated with all the taxa in X.

We can therefore construct a fractal graph in which each vertex is a cluster X, 'branching out' into the clusters $Y_1, Y_2, \ldots$

To express this construction symbolically, let $R(X)$, the 'residue' of the cluster X, be defined to be all the primary taxa, in the system T, which are not in X but are associated with all the taxa in X. Then the cluster X branches out into the cluster Y_1, Y_2, such that

$$Y_k = \{X, p_k\},$$

where

$$R(X) = \{p_1, p_2 \ldots\}.$$

This process terminates if the residue $R(X)$ is empty. Then X is a complete cluster.

For example, if the taxonomic system T consists of the primary taxa p, q, r and their complements $\overline{p}$, $\overline{q}$, $\overline{r}$, then $R(p) = \{q, r, \overline{q}, \overline{r}\}$, and, if $X = p$, Y_1, Y_2, Y_3, Y_4 are

$$\{p, q\}, \quad \{p, r\}, \quad \{p, \overline{q}\}, \quad \{p, \overline{r}\}.$$

Then $R(p, q) = \{r, \overline{r}\}$, and the cluster $\{p, q\}$ branches out into

$$\{p, q, r\}, \quad \{p, q, \overline{r}\}.$$

Finally, $R\{p, q, r\} = \emptyset$, and the taxa $\{p, q, \overline{r}\}\{p, q, r\}$ cannot branch out any further.

The general process starts with the initial (rather degenerate!) clusters, each of which is just one taxon of T. The process of 'branching' may terminate in complete clusters or may proceed indefinitely.

To deal with infinite systems T for which the fractal graph proceeds indefinitely, we introduce the concept of 'paths'.

A path P is a sequence of incomplete clusters such that an (incomplete) cluster X in the path is succeeded by one member of each of the clusters $Y_1, Y_2, \ldots$ which branch out from X.

Thus, in the example quoted above, the sequence

$$\{p\}, \quad \{p, q\}, \quad \{p, q, r\}$$

is a path.

Whether the path P terminates or not we construct the collection Ω consisting of all the taxa in all the cluster in the path P. We prove that Ω is a complete cluster.

If p, q are any two taxa in Ω, then there must be some cluster X on the path P such that, $p \in X$, $q \in X$. Hence $p \sim q$.

Similarly, if x is any taxon not in Ω, then x is excluded from the cluster X, and $x \not\sim p$, $x \not\sim q$.

Thus Ω is a complete cluster.

(*d*) *Atomic taxa*

The constructive definition of complete clusters enables us to define the atomic taxon α derived from a complete cluster A, in the form

$$(\alpha, s) = \prod (p, s),$$

where the product is extended over all primary taxa p and their complements $\overline{p}$.

Hence, $\alpha \sim s$ if $s \in A$ and $\alpha \subset p$ if $p \subset A$.

In order to compare atomic taxa in the taxonomic theory with the 'elements' of a classical set, we establish the main properties of atomic taxa as follows:

(i) If A and B are distinct, complete clusters, and α, β are the corresponding atomic taxa, then there is a taxon x in A and a taxon y in B such that $x \not\sim y$.

For, if A and B are different, then there must be a taxon x in A which is not in B, (or vice-versa). Since x is not in B and B is a complete cluster, then there must be a taxon y in B, such that $y \not\sim x$.

(ii) If A and B are distinct, complete clusters, and α, β are the corresponding atomic taxa, then α and β are dissociated.

For the product for (α, s) contains a factor (x, s), and the product for (β, s) contains a factor (y, s), such that $x \not\sim y$.

Now $(\alpha\beta, s) = \prod(p, s) \cdot \prod(q, s)$, where $p \subset A$ and $q \subset B$.

Hence the product for $(\alpha\beta, s)$ contains the factor $(x, s) \cdot (y, s)$ which is equal to (xy, s). But x and y are dissociated. Therefore by result (v) of § 2 f, $xy = \emptyset$. Hence $(\alpha\beta, s)$ contains the factor $(\emptyset, s)$ and, therefore

$$\alpha\beta = \emptyset.$$

Thus all the atomic taxa in a taxonomic system are dissociated from one another.

(iii) It remains to prove that a taxon p is the sum of all atomic taxa α included in p.

Define a taxon x by the sum

$$(x, s) = \Sigma(\alpha, s)$$

for all the atomic taxa α which are associated with the taxon p. Then $\alpha \sim p$, and $x \sim s$ only if there is an atomic taxon α associated with s. Thus p and s will be in the same complete cluster A, and therefore $p \sim s$, i.e.

$$x \sim s \quad \text{if} \quad p \sim s.$$

Similarly, we can prove that

$$x \not\sim s \quad \text{if} \quad p \not\sim s.$$

Hence, x and p must be identified, and

$$(p, s) = \Sigma(\alpha, s)$$

for all the atomic taxa α such that $\alpha \sim p$.

This establishes the third result, and, incidentally proves that not all these taxa are null. Some of them may be null, and we therefore restrict the term 'atom' to the non-null atomic taxa.

(e) *Atoms and elements of sets*

Hitherto the algebraic elements, (p, s) have been defined by reference to the primary taxa s. But, to exhibit the relations between taxonomic theory and set theory, we now introduce the algebraic element, (p, α), to which the basic reference is to the atomic taxa α and $(p, \alpha) = \mathbf{1}$ if $p \sim \alpha$, i.e. if p is included in the complete cluster A of the atomic taxon α.

Then the union, intersection and complement of taxa can be shown in the form

$$(p, \alpha) + (q, \alpha) = (\rho, \alpha), \quad (p, \alpha) \cdot (q, \alpha) = (\sigma, \alpha),$$

$$(p,\alpha)+(\overline{p},\alpha)=\mathbf{1},\quad (p,\alpha)\cdot(\overline{p},\alpha)=\mathbf{0}.$$

For $(p,\alpha)+(q,\alpha)=\mathbf{1}$ if $\alpha\subset p$, or $\alpha\subset q$, and $(p,\alpha)=\mathbf{1}$ if $\alpha=\rho$ and if $\alpha\subset p$ or $\alpha\subset q$.

Hence $(p,\alpha)+(q,\alpha)=(\rho,\alpha)$. Similarly, we can deduce the other relations given above.

It thus appears that any taxon p is the union of all the atoms α included in p, that any union $\rho=p+q$, is the union of all the atoms included in p or q. Similarly, any intersection $\sigma=pq$, is the intersection of all the atoms included in both p and q.

Thus the atoms of taxonomic theory correspond to the elements of set theory.

5. Propositional logic

(a) *The problems*

Three outstanding problems of propositional logic are (i) what is a proposition, (ii) what is the meaning of negation, and (iii) what is the definition of implication. Taxonomic theory is, perhaps, a rather restricted domain for the investigation of these matters, but it does provide a 'model' which is sufficient to establish the consistency of the usual rules of inference.

Such a model is suggested by the analogy of union and intersection in set theory with conjunction and disjunction respectively in propositional logic. We shall complete the analogy by taking inference to be a species of inclusion, and by introducing a concept of 'compatibility' to correspond to the relation of association.

By suggesting a definition of the compatibility of propositions, we develop the work of Pearce, Sheffer and Nicol, who derived the definition of the logical connections from a single, undefined relation, usually represented by the 'stroke' symbol $p\mid q$. In contrast to these theories, the taxonomic theory of propositions is independent of the concept of 'truth', and relies entirely on the concept of compatibility. It is, in fact, a development of Wittgenstein's definition of implication in terms of 'elementary propositions'.

(b) *Compatibility*

In taxonomic theory a proposition is defined to be a relation of the form, 'the taxa p and q are associated' or 'the taxa p and q are dissociated', and is expressed in the form $p\sim q$ or $p\nsim q$.

The definition of the compatibility of two propositions may seem a matter of some difficulty, but, if we restrict ourselves to a definition in terms of the taxonomic relations, we are led to define two propositions to be compatible if they are relevant and consistent, i.e. if

(i) they refer to one and the same taxon, i.e. if they are of the form

$$p\sim x,\quad p\sim y,$$

and

(ii) the information they provide is consistent, i.e. if $x\sim y$.

(c) *The taxonomic analogy*

Once we have agreed upon a definition of 'compatibility' we can proceed to develop a propositional theory as an exact analogue of the taxonomic theory.

Using upper-case letters, $P, Q,\ldots$ for propositions we can express the compatibility (or incompatibility) of P and Q in the form $P\sim Q$ or $P\nsim Q$.

We can introduce an 'order' into the universe of the *pairs* of propositions by defining that a compatible pair (A, B) 'precedes' an incompatible pair (X, Y), i.e.

$$(A, B) \geqslant (X, Y) \quad \text{if} \quad A \sim B \quad \text{and} \quad X \not\sim Y.$$

Then we can immediately introduce the concept of 'implication', i.e. the analogue of 'inclusion' by writing $P \Rightarrow Q$ if P implies Q and by defining this relation as

$$P \Rightarrow Q \quad \text{if} \quad (P, S) \leqslant (Q, S),$$

i.e. P implies Q if any proposition S compatible with P is also compatible with Q.

A collection of 'primary' propositions can be defined by the condition that no proposition in the collection is implied by any other proposition in the collection.

The 'maximum' and 'minimum' of a collection of pairs (A, S) of primary propositions can be defined as for pairs of taxa (§ 2 c), and then we can define the conjunction and disjunction in the form

$$X = A \cup B \quad \text{if} \quad (X, S) = \max\{(A, S), (B, S)\},$$

$$Y = A \wedge B \quad \text{if} \quad (Y, S) = \min\{(A, S), (B, S)\}.$$

As usual, we shall write

$$X = A + B \quad \text{and} \quad Y = A \times B.$$

The analogue of the 'complement' $\overline{p}$ of a taxon p is the 'negative' A^* of the proposition A, and is defined, by analogy with the complement, in the form

$$A \not\sim A^*$$

and

$$A^* \not\sim S \quad \text{if} \quad A \sim S,$$

for all primary propositions S.

To interpret the definition of 'implication' we can regard the set of primary propositions S, which are compatible with A, as the 'evidence' for A is included in the evidence for B.

The 'null' proposition $\emptyset$ can be defined as a proposition incompatible with any other proposition, P, whence

$$\emptyset P = \emptyset \quad \text{and} \quad \emptyset \Rightarrow P.$$

(d) *Rules of inference*

The fundamental rules of inference are the analogues of the formulae which express the relation of inclusion.

Thus, since

$$(P \cdot Q, S) = (P, S) \cdot (Q, S),$$

it follows that $P \Rightarrow Q$, if and only if

$$P = P \cdot Q.$$

Also, since

$$(P + Q, S) = (P, S) + (Q, S),$$

an equivalent relation is

$$Q = P + Q.$$

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To show that, if $Q \Rightarrow R$, then

$$P \cdot Q \Rightarrow P \cdot R,$$

we note that, if $Q \Rightarrow R$, then

$$Q = Q \cdot R$$

and

$$(P \cdot R)(Q \cdot R) = (P \cdot Q) \cdot R = P \cdot (Q \cdot R) = P \cdot Q,$$

whence

$$(P \cdot Q)(P \cdot R) = (P \cdot R)(Q \cdot R) = (P \cdot R)Q = P \cdot Q,$$

and

$$P \cdot Q \Rightarrow P \cdot R.$$

The rule of the syllogism, namely that if $X \Rightarrow Y$ and $Y \Rightarrow Z$, then $X \Rightarrow Z$, follows from the relations

$$X = X \cdot Y = X(Y \cdot Z) = X \cdot Z.$$

Finally we note that, since $P \Rightarrow Q$ is equivalent to $P = PQ$, it is also equivalent to $Q = P \cup Q$, to

$$PQ^* = \emptyset$$

and to

$$P^* \cup Q = \cup(P^*, P, Q) = T$$

the 'total' or universal proposition, which is implied by any proposition, for $X = XT$ for all X.

The last results must be contrasted with the Philonian proposition $P^* \cup Q$ for propositions P, Q which may take the values 'true' or 'false'. The Philonian proposition by itself is not sufficient to define implication, as indeed, was noticed by Frege. In taxonomic theory, the reason for this assertion is that $P^* \cup Q$, by itself, is a proposition about taxa, while $P \Rightarrow Q$ is a proposition about propositions, and is of a higher 'type'.

6. Abstract arithmetic

(*a*) *The main problems*

The main problems of abstract arithmetic are the definition of cardinal integers (hereafter referred to as 'numbers') and the theory of sequences of numbers. The best-known account of this subject is undoubtedly that due to Bertrand Russell (Russell 1903), but, in spite of its clarity and simplicity, his account does need to be completed by certain fundamental concepts and theorems.

In Russell's theory, the 'number' of a collection C of objects is defined to be the whole class of collections which are 'similar' to C, and 'similarity' is defined by means of a symmetric, one-to-one relation between two collections. Moreover, he defines 'finite' numbers to be numbers for which the principle of mathematical induction is valid. It is our purpose to provide a definition of the relation of similarity and a proof of the principle of mathematical induction.

(b) Counting

In defining a number, we are confronted with Russell's paradox again, for a number is indubitably the number of a 'collection' (not the number of a taxon), and we have to avoid 'rogue' collections which are members of themselves.

In taxonomic theory this difficulty is avoided by restricting ourselves to collections of 'primary' taxa as defined in § 1 *e*. A primary taxon cannot be included in any other primary taxon and a collection of primary taxa is not itself a primary taxon. Thus Russell's paradox is eliminated.

It is the possibility of defining primary taxa which is the main contribution of taxonomic theory to abstract arithmetic. The main instrument in the theory is not the theory of series but the more general and powerful method of 'fractal analysis', which enables us to discuss mathematical systems in which each element has, not one, but many 'successors'. This enables us to provide a theory of 'counting' and a definition of 'similarity'.

If we attempt to analyse the concept of 'counting' a collection of items, it appears that the operation applies only to selections which can be arranged in some serial order, like a flock of sheep leaving the fold through a gate. The counting then consists of correlating such an ordered collection with some standard collection consisting of distinct and recognizable signs, such as the letters of the alphabet or the notches on a tally-stick.

But, if a collection is well ordered in one way, it can be well ordered in many ways, which are simply permutations of one another. A theory of counting must, therefore, take into account all possible ways of counting an assembly, and thereby avoid the necessity of making successive sets of choice, and thus prove the consistency of the counts of an assembly when arranged in different orders.

This leads us to introduce a 'graded' fractal diagram in which the vertices are arranged in consecutive 'grades', while each vertex in a grade G is connected by directed branches to the many vertices in the successive grade H.

(c) The fractal process

Our purpose is to construct a complete set of all the 'ordered' collections of primary taxa, which are 'equivalent' to a given finite collection C. To do this we classify all the sub-collections of C in a sequence of 'grades', all the sub-collections in any grade being ordered and similar.

The initial grade consists of all the unary sub-collections in C, so that each of these is a single taxon P_n.

The succeeding grade consists of all the binary sub-collections in C, such as $\{p_m, p_n\}$; and each unary sub-collection $\{p_j\}$ in the initial grade is the predecessor of all the binary sub-collections $\{p_j, p_k\}$ (for all taxa p_k except p_j).

In general, $G_1, G_2, \ldots, G_k$ are the sub-collections in a grade G; the sub-collections in the succeeding grade H are $H_{m,n}$ where

$$H_{m,n} = G_m + p_n,$$

for all taxa p_n except those in G_m.

Thus the additional taxa p_n are not picked out by any arbitrary choice but by taking all the taxa not in G_m, one at a time.

The sub-collection $H_{m,n}$ are all ordered (and indeed well ordered) since the taxon p_n succeeds all the taxa in G_m.

The terminal grade contains the ordered collections, $C_1, C_2, \ldots$, each or which consists of all the taxa in the original collection C.

We now define the 'number' of taxa in C to be the set C^* of the ordered collection $\{C_1, C_2, \ldots\}$, each of which is a 'version' of C.

The process by which such C_k is constructed is clearly a species of 'counting' the collection C, and the process establishes what has been called the 'fundamental theorem of arithmetic', i.e. that all 'counts' of C must give the same total number.

(*d*) *Comparability of numbers*

To compare the numbers, $n(A)$ and $n(B)$, of two disjoint collections A and B, we construct two fractal diagrams similar to the diagram constructed in § 6 *c*.

In the initial grade of each diagram the entries are the binary sub-collection A_1, containing all taxa p in A, and B_1, containing all the taxa q in B.

If $\{A_m\}$ and $\{B_m\}$ are the sub-collections of A and B represented in the corresponding grades G_A and G_B, then the corresponding sub-collections $\{A_{m,n}\}$ and $\{B_{m,n}\}$ in the succeeding grades H_A and H_B, are given as

$$A_{m,n} = A_m + p_n, \quad B_{m,n} = B_n + q_n,$$

where p_n is in A but not in A_m, while q_n is in B but not in B_m.

Each diagram terminates when $A_{m,n} = A_m$ or $B_{m,n} = B_n$. The numbers of taxa in A and B are

$$n(A) = \{A_1^*, A_2^* \ldots\}, \quad n(B) = \{B_1^*, B_2^* \ldots\},$$

the asterisked collection denoting the terminal collection in the diagram of A and B respectively. There terminal collections may, or may not, belong to corresponding grades.

If $\{A'_1, A'_2 \ldots\}$ is the terminal 'A grade' and the corresponding B grade is *not* the terminal B grade, then we define that $n(A) > n(B)$ and vice-versa.

If the terminal A grade does correspond to the terminal B grade then we define that $n(A) = n(B)$.

(*e*) *The Peano series*

The set of entities which we have constructed as the 'numbers' $n(A)$ and $n(B), \ldots$, of collections $A, B, \ldots$ correspond to the numbers characterized by the first four of the axioms due to Peano.

The unit number, 1, is the number $n(U)$ of any unary collection $U = \{p\}$.

If a collection B is the sum of a collection A and any taxon not in A, then the number $n(B)$ is the successor of the number $n(A)$, so that

$$p(B) = n(A) + 1 \quad \text{if} \quad B = A + p \quad \text{and} \quad p \not\subset A.$$

No two numbers can have the same successor, and 1 is not the successor of any number.

It remains to show that the set of the taxa, $n(A)$, satisfies the principle of mathematical induction.

(*f*) *Mathematical induction*

The principle of mathematical induction is a property of an inductive sequence of propositions, i.e. of a collection C of propositions such that to any number λ there corresponds a proposition p_λ of C, such that if μ is the successor of λ, i.e. $\lambda + 1$, then

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p_λ implies p_μ. Thus

$$p_\lambda \Rightarrow p_\mu$$

and

$$p_\lambda = p_\lambda p_\mu.$$

The principle of mathematical induction can then be put in the form

$$p_1 \Rightarrow p_\lambda, \quad \text{for all} \quad \lambda,$$

i.e. the initial proposition p_1 implies each proposition of the inductive sequence.

To prove this principle we consider all the implications

$$p_1 = p_1 p_2, \quad p_2 = p_2 p_3, \quad \text{and} \quad p_\lambda = p_\lambda p_\mu;$$

and we form the intersection

$$q_\lambda = p_1 p_2 \ldots p_\lambda.$$

Then

$$q_\lambda = q_\mu,$$

whence

$$q_1 = q_2 \ldots = q_\lambda = q_\mu.$$

From this we deduce that

$$p_1 p_n = q_1 p_n = q_n p_n = (p_1 p_1 \ldots p_n) p_n = q_n = p_1.$$

Therefore $p_1 = p_1 p_n$ and p_1 implies p_n.

We note that this proof depends upon the transitivity of the relation of equivalence ($q_\lambda = q_\mu$) in propositions.

7. Type theory

(a) *Classical type theory*

Although our studies of taxonomic theory have been mostly restricted to taxa of one and the same system, the definition of 'power systems' (§ 1 *j*) has required the simultaneous study of many systems closely related to one another. In this chapter we shall describe species of 'type theory' which treats this topic in general.

We take as our example the different species of numbers – signless integers, signed integers, rational numbers, real numbers – regarding these as if they were various kinds of taxonomic systems. The two relevant characteristics of these different systems are that they are essentially disjoint but capable of assimilation.

Thus the signed integers, $\pm 1, \pm 2, \ldots$ are not a subset of the rational numbers but a distinct system of numbers and the signed integers and the rational numbers are of different 'types'. Nevertheless the signed integers form a collection which is isomorphic to a certain subset of rational numbers, namely the rational numbers of the form $\frac{1}{1}, \frac{2}{1}, \ldots$. This 'assimilation' of the integer p to the rational number $p/1$, enables us to pass readily from an equation

$$qx = p$$

involving integers to the solution

$$x = p/q$$

involving rational numbers.

Any relation between integers can be replaced by the 'isomorphic' equation which involves rational numbers. There is no need for any 'axiom of reducibility'. Instead we have a theorem of 'elevation' in the form of the morphism

$$p \rightarrow p/1.$$

We shall use this 'classical' type theory to suggest a type theory for propositions.

(*b*) *Comparable types of taxonomic systems*

Without attempting a comprehensive theory of different types of taxonomic system, we can consider the theory of 'comparable' types, which we defined as follows.

The taxonomic systems T and S are of 'comparable' type if all the taxa of the system T are isomorphic with the primary taxa of the system S. This implies that the taxa of T have the same taxonomic relations to one another as primary taxa of S as they have in T, and, in addition, have the appropriate relations to the secondary taxa of S. Thus, if we denote the taxonomic relations of the taxa p, q in T and of the taxa x, y in S by the symbols

$$(p, q)_T \quad \text{and} \quad (x, y)_S$$

respectively, then

$$(p, q)_S = (p, q)_T \quad \text{for all} \quad p, q \in T,$$

but the taxa of T also acquire the relations

$$(p, x) \quad \text{for} \quad p \in T, \quad x \in S.$$

Hence to any formula involving the taxa of T there corresponds the same formula involving the primary taxa of S. There is no need for an axiom of reducibility to 'degrade' the taxa of S into taxa of T. Instead we have promoted the taxa of T by transforming them into taxa of S.

The simplest examples are provided by making all the clusters of T to be the primary taxa of S, or by making all the neighbourhoods $N(p)$ of the taxa of T like the primary taxa of S.

(*c*) *Philonan inference*

The distinction between different types of proposition is important in connection with the definition of implication.

The propositions $p \sim q$ and $p \not\sim q$ make assertions about the taxa p and q and their relations. The proposition $P \Rightarrow Q$ makes an assertion about the propositions P and Q. Hence the implication, $P \Rightarrow Q$, is a proposition of a different type from the proposition P or Q. Also the proposition $P^* \cup Q$ makes no assertion about P or Q, but only about the taxa to which P and Q refer.

Hence the Philonan proposition $P^* \cup Q$ cannot define implication in our taxonomic system.

Even in classical propositional logic this proposition plays a dubious role, for it is easily verified that it is inconsistent with the usual rules of inference.

For example, a standard rule of inference is

$$XY \Rightarrow X.$$

But if we put $XY = P, X = Q$, the Philonic proposition becomes

$$P^* \cup Q = (XY)^* \cup Y = (X^* \cup Y^*) \cup Y = Y \cup Y^* = T,$$

the universal proposition.

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In practice it seems that in the classical theory, it is better to take implication as an undefined relation, governed by the rules of inference.

(d) *Russell's paradox*

Finally we note that it is not necessary to invoke our taxonomic type theory in order to dispose of Russell's famous paradox.

A taxonomic system is an entity of a different order from any of its constituent taxa, simply because we cannot prescribe any taxonomic relation, either of association or dissociation between a system T and one of its taxa p.

What we have done (in § 1 h) is to define a 'total' taxon τ which is associated with all primary taxa in T, and thereby becomes a secondary taxon of T.

What we cannot do is to give a constructive definition of a 'super-system' Z, whose primary taxa are all the systems T – for we have no means of prescribing relations of association between systems.

8. Conclusion

The investigations of this paper are very far from providing fundamental theories for the whole of contemporary mathematics. What they have achieved is a new method of constructing fundamental theories, and the results of this method when applied to set theory, arithmetic and propositional logic. Algebra and geometry have not been considered, nor Gödel's theorems on self-reflexive propositions.

The purpose of Hilbert's work on the foundations of mathematics has been described by the Kneales who maintain that it requires

> a system which is sufficient for the derivation of number theory, classical analysis, and the theory of sets must contain not only the rules and axioms of the restricted calculus of propositional functions, but also Frege's axioms of identity, axioms of number like those of Peano, and axioms of membership like those of Zermelo.

This summarizes what we have attempted in this paper as a contribution to what Fraenkel has described as 'the supreme problem of mathematics'.

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